Received: *June 27, 2001,* Accepted: *August 3, 2001*

Split string field theory I

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Abstract: We describe projection operators in the matter sector of Witten's cubic string field theory using modes on the right and left halves of the string. These projection operators represent a step towards an analytic solution of the equations of motion of the full string field theory, and can be used to construct Dp-brane solutions of the string field theory when the BRST operator Q is taken to be pure ghost, as suggested in the recent conjecture by Rastelli, Sen and Zwiebach. We show that a family of solutions related to the sliver state are rank one projection operators on the appropriate space of half-string functionals, and we construct higher rank projection operators corresponding to configurations of multiple D-branes.

Keywords: Bosonic Strings, D-branes, Spontaneous Symmetry Breaking.

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Contents

1. Introduction

In the last 18 months, renewed attention has been paid to Witten's cubic bosonic open string field theory [1], following Sen's conjectures that this formalism can be used to give an analytic description of D25-brane decay in bosonic string theory [2]. While much progress has been made towards proving Sen's conjectures using the level truncation approximation [3], the large B field simplification [4], background independent string field theory [5], and other methods [6] little progress has been made towards finding a nontrivial analytic solution of the cubic string field theory equations of motion

$$Q\Phi = -\Phi \star \Phi \,. \tag{1.1}$$

It was pointed out by Witten in [1] that the string field star algebra is roughly describable as the algebra of operators acting on a space of states associated with a half string. In such a formalism, the dependence of the string field on the degrees of freedom living on the right and left halves of the string are separated, so the string field can be written in the form $\Psi[l, r]$. In the operator formalism the star product, which can be written as $(\Psi \star \Phi)[l, r] = \int \mathcal{D}x \Psi[l, x]\Phi[x, r]$ simply becomes multiplication of operators. Making such an operator formalism for the string field precise is a difficult problem which has not yet been solved. Midpoint insertions in the ghost sector and the structure of the BRST operator present particular complications. In this paper we use a formalism of this type to perform calculations in the matter sector of the string field theory, where there are fewer subtleties involved. We describe states in the string field theory in terms of modes on the left and right halves of a split string. We use this formalism to identify solutions of the projection equation

$$\Psi_m = \Psi_m \star \Psi_m \tag{1.2}$$

for string fields in the matter sector of the string field theory in 26-dimensional flat space. This work represents a first step towards developing a split string formalism for the full string field theory which could be used to rewrite and perhaps to solve the full string field theory equation of motion (1.1). The techniques we use in this paper can be extended to describe the ghost sector of the theory, although there are a number of additional complications in this case; the details of the analysis in the ghost sector will be described in a later publication [7].

Recently it was suggested by Rastelli, Sen and Zwiebach (henceforth RSZ) that in the locally stable vacuum in which the space-filling D25-brane has been annihilated, it may be possible to describe the shifted string field theory using a BRST operator which is pure ghost [8]. With a pure ghost BRST operator of this kind, the equations of motion (1.1) can be factorized into a ghost part Ψ_g satisfying (1.1) and a matter part Ψ_m satisfying (1.2), when the full string field takes the form $\Phi = \Psi_m \otimes \Psi_g$. This factorization was used in [9] to identify a certain solution of (1.2) found in [10, 11] as the matter part of a D25-brane solution. Even if the RSZ conjecture is not true, the BRST operators these authors suggest define a new interesting class of string field theories. In [8], a particular class of candidates for the pure ghost Q were suggested, namely the operators $Q = \sum a_n(c_n + (-1)^n c_{-n})$. These operators all have trivial cohomology and satisfy $Q^2 = 0$, both properties which are expected of the BRST operator in the closed string vacuum. In addition to these properties, there are two additional properties which should be expected of the BRST operator: first, that $\int Q\Psi = 0 \ \forall \Psi$ (equivalent to the condition $QI = 0$ where I is the identity of the string field star algebra), and second that Q act as a derivation of the star algebra. Not all the operators in the class suggested by RSZ satisfy both of these conditions, which may give some clues as to the correct choice of operator.

In the half-string formalism we consider here, the solutions of (1.2) are simply projection operators onto half-string states, which can be described as functionals of the coordinate fields on the half string. A simple class of rank one projection operators take the form

$$e^{-\frac{1}{2} l \cdot M \cdot l - \frac{1}{2} r \cdot M \cdot r}\,, \tag{1.3}$$

where l, r describe the degrees of freedom of the left and right halves of the string. We show that the D-instanton state related to the sliver state found in [10, 11, 9] is a projection operator of this form. We show that this state can be modified in various ways to produce other projection operators describing the D-instanton and higher-dimensional Dp-branes. Rank r projection operators given by the sum of r orthogonal rank one projection operators can be identified with solutions of (1.2) describing r D-branes.

In section 2 we develop the half-string formalism and describe the structure of the cubic string field theory in this language. In section 3 we describe projection operators using half-string degrees of freedom. We derive a condition which must be satisfied by a gaussian state in the string Fock space for it to have the form of a rank one projector onto a gaussian half-string state. We use the D-instanton state, which is a rank one projection of this form, to construct a class of projectors describing Dp-branes of arbitrary dimension, and we discuss the construction of higher rank projectors corresponding to multiple D-branes. Section 4 contains some concluding remarks. The appendix contains a proof that the D-instanton state is a rank one projection. After this work was completed we learned of related work by Rastelli, Sen and Zwiebach [12].

2. Half-string formalism

In this section we develop the formalism to describe string states in terms of half-string degrees of freedom. This formalism was implicitly used to construct the three-string vertex operator in [13]. A related formalism was further developed in [14, 15]. We will only describe the matter fields $X^i, i \in \{0, 1, \ldots, 25\}$ here; a similar approach can be used for the ghost fields, which will be presented in detail elsewhere. For the most part, we will use the notation and conventions of [13].

2.1 Matter fields

The bosonic string has 26 matter fields X^i. We can expand each of these fields in modes through

$$x(\sigma) = x_0 + \sqrt{2} \sum_{n=1}^{\infty} x_n \cos(n\sigma) \,. \tag{2.1}$$

(We drop spatial indices in most formulae for clarity.) We can relate the modes in (2.1) to creation and annihilation operators through

$$\begin{aligned} x_n &= \frac{i}{\sqrt{2n}} \left(a_n - a_n^\dagger\right) & \qquad p_n &= -i\frac{\partial}{\partial x_n} = \sqrt{\frac{n}{2}} \left(a_n + a_n^\dagger\right) \\ a_n &= -i\left(\sqrt{\frac{n}{2}} x_n + \frac{1}{\sqrt{2n}} \frac{\partial}{\partial x_n}\right) & \qquad a_n^\dagger &= i\left(\sqrt{\frac{n}{2}} x_n - \frac{1}{\sqrt{2n}} \frac{\partial}{\partial x_n}\right), \end{aligned} \tag{2.2}$$

for $n \neq 0$, and through

$$x_0 = \frac{i}{2}\left(a_0 - a_0^\dagger\right) \qquad p_0 = -i\frac{\partial}{\partial x_0} = \left(a_0 + a_0^\dagger\right)$$
$$a_0 = -i\left(x_0 + \frac{1}{2}\frac{\partial}{\partial x_0}\right) \qquad a_0^\dagger = i\left(x_0 - \frac{1}{2}\frac{\partial}{\partial x_0}\right), \tag{2.3}$$

for the zero modes. We write

$$|x) = \frac{i}{\sqrt{2}}E[|a) - |a^\dagger)]$$
$$|p) = \frac{1}{\sqrt{2}E}[|a) + |a^\dagger)]\,, \tag{2.4}$$

where

$$[a_n, a_m^\dagger] = \delta_{nm}\,, \qquad E^{-1}_{nm} = \delta_{nm}\sqrt{n} + \delta_{n0}\delta_{m0}\sqrt{2}\,. \tag{2.5}$$

A state in Fock space can be described as a linear combination of Fock space states of the form $|\Psi\rangle = \prod a_1^{\dagger\,i_1} a_2^{\dagger\,i_2} \cdots a_n^{\dagger\,i_n} \cdots |\Omega\rangle$, or as a functional on the space of string configurations

$$\begin{aligned}\Psi\left[x(\sigma)\right] &= \langle x(\sigma)|\Psi\rangle \\ &= \langle\Omega|\exp\sum_n\left[-\frac{1}{2}nx_nx_n - x_0^2 + i\sqrt{2n}a_nx_n + 2ia_0x_0 + \frac{1}{2}a_na_n\right]|\Psi\rangle\end{aligned} \tag{2.6}$$

where

$$|x(\sigma)\rangle = \exp\left[-\frac{1}{2}(x|E^{-2}|x) - i(a^\dagger|\sqrt{2}E^{-1}|x) + \frac{1}{2}(a^\dagger|a^\dagger)\right]|\Omega\rangle\,. \tag{2.7}$$

2.2 Elements of string field theory

The string field Ψ is a functional, which can be expressed as above either in Fock space language or as $\Psi[\{x_n\}]$ in terms of the string modes as in (2.6).

The star product of two string fields Ψ_1, Ψ_2 can be written as the functional [1]

$$\begin{aligned}\left(\Psi_1 \star \Psi_2\right)[z(\sigma)] &\equiv \int \Psi_1[x(\sigma)]\Psi_2[y(\sigma)] \prod_{\frac{\pi}{2}\leq\sigma\leq\pi} \delta[x(\sigma) - y(\pi-\sigma)] \prod_{\frac{\pi}{2}\leq\sigma\leq\pi} dx(\sigma) \prod_{0\leq\sigma\leq\frac{\pi}{2}} dy(\sigma) \\ z(\sigma) &= x(\sigma) \qquad \text{for } 0 \leq \sigma \leq \frac{\pi}{2}\,, \\ z(\sigma) &= y(\sigma) \qquad \text{for } \frac{\pi}{2} \leq \sigma \leq \pi\,.\end{aligned} \tag{2.8}$$

While the infinite product of delta functions over a finite interval in (2.8) may seem to only have formal meaning, this expression is given a precise meaning in terms of Fourier modes on the string. This approach was used in [13] to construct a Fock space representation of the 3-string vertex.

The identity with respect to the star product $\star$ is the state

$$I[x(\sigma)] \equiv \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta[x(\sigma)-x(\pi-\sigma)] = \prod_n \delta(x_{2n+1})\,. \tag{2.9}$$

This state can be written in Fock space language as [13]

$$|I\rangle = \exp\left[-\frac{1}{2}(a^\dagger|C|a^\dagger)\right]|\Omega\rangle = \exp\left[-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n a_n^\dagger a_n^\dagger|\right]|\Omega\rangle\,, \tag{2.10}$$

where C is given by

$$C_{nm} = \delta_{nm}(-1)^n\,. \tag{2.11}$$

The integral of a string field Ψ is defined by

$$\int \Psi = \langle\Psi|I\rangle\,, \tag{2.12}$$

where $\langle\Psi|$ is the hermitean conjugate (not the BPZ dual) of $|\Psi\rangle$. For a pair of string fields Ψ_1, Ψ_2 the integral of $\Psi_1\star\Psi_2$ is given by

$$\int \Psi_1\star\Psi_2 = \int \Psi_1[x(\sigma)]\Psi_2[y(\sigma)] \prod_{0\leq\sigma\leq\pi} \delta[x(\sigma)-y(\pi-\sigma)]dx(\sigma)dy(\sigma)\,. \tag{2.13}$$

The string field theory action (including ghosts) is given by

$$S = -\int\left[\frac{1}{2}\Psi\star Q\Psi + \frac{1}{3}\Psi\star\Psi\star\Psi\right]. \tag{2.14}$$

The equation of motion from this action is

$$Q\Psi = -\Psi\star\Psi\,. \tag{2.15}$$

The standard string BRST operator Q mixes the matter and ghost sectors, so that solving (2.15) involves a highly nontrivial interplay between the matter and ghost sectors. As mentioned above, in the RSZ model the operator Q is taken to be pure ghost. For such a BRST operator, and for states of the form $\Psi = \Psi_m\otimes\Psi_g$, the equations of motion factorize as discussed in the Introduction and in the matter sector we simply need to solve the projection equation

$$\Psi_m = \Psi_m\star\Psi_m\,. \tag{2.16}$$

Even if the conjecture of Rastelli, Sen and Zwiebach is incorrect, it is still of interest to find solutions to (2.16). Solving equations of this form has led to substantial insights into the form of solitons describing lower-dimensional Dp-branes in the large B limit and in noncommutative geometry (see [16] for a review and further references). It seems likely that developing a better understanding of solutions to (2.15) and (2.16) separately in the ghost and matter sectors will lead to better tools for attacking the problem of solving the full equation of motion in the original theory where Q couples the matter and ghost sectors. In the remainder of this paper, we focus on finding solutions to (2.16) in the matter sector.

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2.3 Splitting the string

In this section we describe the transformations needed to rewrite the string field in terms of modes on the right and left parts of the string. We break up the string coordinate $x(\sigma)$, which satisfies Neumann boundary conditions at $\sigma = 0,\ \pi$, into its left and right pieces, according to

$$l(\sigma) = x(\sigma), \quad r(\sigma) = x(\pi - \sigma) \qquad \text{for } 0 \leq \sigma \leq \frac{\pi}{2}\,, \tag{2.17}$$

where $l(\sigma)$ and $r(\sigma)$ obey Neumann boundary conditions at $\sigma = 0$ and Dirichlet boundary conditions at $\sigma = \pi/2$. We can perform a separate mode expansion on the left and right pieces of the string

$$l(\sigma) = \sqrt{2}\sum_{n=0}^{\infty} l_{2n+1}\cos(2n+1)\sigma\,, \tag{2.18}$$

$$r(\sigma) = \sqrt{2}\sum_{n=0}^{\infty} r_{2n+1}\cos(2n+1)\sigma\,. \tag{2.19}$$

Note that the functions $l(\sigma), r(\sigma)$ defined through (2.18), (2.19) need not vanish at the point $\sigma = \pi/2$, since for example if $l_{2n+1} = \frac{2\sqrt{2}(-1)^n}{(2n+1)\pi}a + o(1/n)$, then using

$$\sum_n \frac{4(-1)^n}{(2n+1)\pi}\cos(2n+1)\sigma = 1 \qquad \text{for } 0 < \sigma < \frac{\pi}{2}\,,$$

we have $l(\pi/2) = a$. If $x(\sigma)$ is a well-behaved (smooth) function then $l(\sigma)$, $r(\sigma)$ defined through (2.17) will take the value $x(\pi/2)$ at $\sigma = \pi/2$.

Using

$$\int_0^{\frac{\pi}{2}} d\sigma \cos 2n\sigma \cos(2m+1)\sigma = \begin{cases} \frac{\pi}{4}X_{2n,2m+1} & (\text{for } n \neq 0) \\ \frac{\pi}{2\sqrt{2}}X_{0,2m+1} & (\text{for } n = 0)\,, \end{cases} \tag{2.20}$$

where

$$\begin{aligned} X_{2k+1,2n} = X_{2n,2k+1} &= \frac{4(-1)^{k+n}(2k+1)}{\pi\left((2k+1)^2 - 4n^2\right)} \qquad (n \neq 0)\,, \\ X_{2k+1,0} = X_{0,2k+1} &= \frac{2\sqrt{2}(-1)^k}{\pi(2k+1)}\,, \end{aligned} \tag{2.21}$$

we can relate the full-string modes to the half-string modes through

$$\begin{aligned} x_{2n+1} &= \frac{1}{2}\left(l_{2n+1} - r_{2n+1}\right), \\ x_{2n} &= \frac{1}{2}\sum_{k=0}^{\infty} X_{2n,2k+1}\left(l_{2k+1} + r_{2k+1}\right). \end{aligned} \tag{2.22}$$

The matrix

$$X \equiv \begin{pmatrix} 0 & X_{2k+1,2n} \\ X_{2n,2k+1} & 0 \end{pmatrix} \tag{2.23}$$

is symmetric and orthogonal: $X = X^T = X^{-1}$. Furthermore, we have

$$\sum_{n=1}^{\infty} X_{2k+1,2n}(-1)^n = -\frac{1}{\sqrt{2}}X_{2k+1,0}\,, \qquad \sum_{n=0}^{\infty} X_{2k+1,2n}(2n)^2 X_{2n,2k'+1} = (2k+1)^2\delta_{kk'}\,. \tag{2.24}$$

Note that the matrix X defined in (2.23) differs from the equivalent X defined in [13] by a factor of i.

We can invert (2.22) to derive

$$\begin{aligned} l_{2k+1} &= x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}x_{2n}\,, \\ r_{2k+1} &= -x_{2k+1} + \sum_{n=0}^{\infty} X_{2k+1,2n}x_{2n}\,. \end{aligned} \tag{2.25}$$

Note that care most be taken when using these formulae to deal with functions which do not satisfy $\sum_n nx_n^2 < \infty, \sum_k k(l_{2k+1}^2 + r_{2k+1}^2) < \infty$. If, for example, we take $l_{2k+1} + r_{2k+1} = X_{2k+1,0} = 4(-1)^k/\pi(2k+1)$, then we must take care with the order of summation when evaluating $x(\pi/2) = \sum(-1)x_{2n}$ since

$$\frac{\sqrt{2}-1}{\sqrt{2}} = \sum_{k=0}^{\infty}\left[\left(\sum_{n=0}^{\infty}(-1)^n X_{2n,2k+1}\right)X_{2k+1,0}\right] \neq \sum_{n=0}^{\infty}\left[(-1)^n\sum_{k=0}^{\infty}\left(X_{2n,2k+1}X_{2k+1,0}\right)\right] = 1. \tag{2.26}$$

2.4 Operator representation of star algebra

The full-string position basis states $|x(\sigma)\rangle$ defined through (2.7) can be related to a half-string position basis through

$$|x(\sigma)\rangle = |\{x_n\}\rangle = |l(\sigma), r(\sigma)\rangle = |\{l_{2k+1}\}, \{r_{2k+1}\}\rangle\,, \tag{2.27}$$

using (2.17), (2.22).

For every state $|\Psi\rangle$ in the Hilbert space there is an associated functional $\Psi[x(\sigma)]$ given by

$$\Psi[x(\sigma)] = \Psi[l(\sigma), r(\sigma)] = \langle l(\sigma), r(\sigma)|\Psi\rangle\,. \tag{2.28}$$

We can associate the functional (2.28) with an operator $\hat{\Psi}$ on the space of functionals of the half-string degrees of freedom, which we write formally as

$$\Psi[x(\sigma)] \Rightarrow \hat{\Psi} = \int \mathcal{D}l\mathcal{D}r\; |l\rangle\Psi[l,r]\langle r|\,, \tag{2.29}$$

or equivalently:

$$\langle x(\sigma)|\Psi\rangle = \langle l|\hat{\Psi}|r\rangle\,. \tag{2.30}$$

These functionals are given precise meaning by using the left and right half-string modes described in section 2.3.

In the language of operators, integration of the string field corresponds to taking the trace of the operator

$$\int \Psi \Rightarrow \mathrm{Tr}[\hat{\Psi}]\,. \tag{2.31}$$

The star product $\star$ is equivalent to the multiplication of operators acting on half-string functionals

$$\Psi_1 \star \Psi_2 \Rightarrow \hat{\Psi}_1\hat{\Psi}_2\,. \tag{2.32}$$

The identity of the star algebra becomes the identity operator on half-string states

$$I[x(\sigma)] = \prod_{0\leq\sigma\leq\frac{\pi}{2}} \delta[l(\sigma)-r(\sigma)] \Rightarrow \hat{I} = \int \mathcal{D}l(\sigma)|l(\sigma)\rangle\langle l(\sigma)|\,. \tag{2.33}$$

For the purposes of this paper we primarily think of the functional $\Psi[l,r]$ associated with the Fock space state $|\Psi\rangle$ as a single operator acting on the space of functionals on the half string. One way of describing the action of this operator is to explicitly separate out the midpoint $x = x(\pi/2) = \sum_n (-1)^n x_{2n}$. We can modify (2.22) and (2.25) by replacing $x_0 \to x_0 - x$, so that $l(\sigma), r(\sigma)$ defined through (2.18), (2.19) satisfy $l(\pi/2) = r(\pi/2) = 0$ when $x(\sigma)$ is smooth. The functional $\Psi[l,r]$ can then be described as an operator-valued function of x, $\hat{\Psi}(x)$, where the operator at each point in space-time acts on a separate copy of the usual Hilbert space associated with a string having Neumann boundary conditions on one end and Dirichlet boundary conditions on the other. In transforming to half-string variables, only x_0 depends upon the midpoint value x, so that a Fock space state with well-defined momentum p_0 becomes an operator-valued functional of the form $e^{ip_0x}\hat{\Psi}$ where $\hat{\Psi}$ acts on the ND string Fock space. This construction can be generalized to an arbitrary space-time manifold $\mathcal{M}$, where $\hat{\Psi}(x)$ becomes an operator-valued function on $\mathcal{M}$. A particularly simple class of operators are those which are independent of the zero mode x_0, and hence of the midpoint x. These operators can be thought of simply as operators acting on the usual Hilbert space of a string having Neumann boundary conditions on one end and Dirichlet boundary conditions on the other end. While the point of view where the midpoint is treated separately becomes helpful in addressing certain problems in the ghost sector of the theory, we will not treat the midpoint separately in most of the equations in this paper. We will mention this point of view again briefly when discussing some points where this perspective is helpful.

2.5 Squeezed states

In this section we collect some formulae which will be useful in discussing projection operators in the split string formalism.

A particularly nice class of states are squeezed states of the form

$$\exp\left(-\frac{1}{2}a_n^\dagger S_{nm} a_m^\dagger\right)|\Omega\rangle\,. \tag{2.34}$$

Up to an overall constant, such a state is described by the functional

$$\exp\left\{-\frac{1}{2}\left(x\left|E^{-1}\left[\frac{1-S}{1+S}\right]E^{-1}\right|x\right)\right\}. \tag{2.35}$$

This relationship can easily be inverted. The functional

$$\exp\left(-\frac{1}{2}\left(x|E^{-1}DE^{-1}|x\right)\right) \tag{2.36}$$

corresponds (up to an overall constant) to the state

$$\exp\left(-\frac{1}{2}\left(a^\dagger\left|\frac{1-D}{1+D}\right|a^\dagger\right)\right)|\Omega\rangle\,. \tag{2.37}$$

3. Projection operators

We will now use the half-string formalism developed in the previous section to describe solutions to the projection equation

$$\Psi = \Psi \star \Psi\,. \tag{3.1}$$

Such solutions may be useful for finding an analytic solution to the full SFT equations, as discussed for example by Kostelecky and Potting [10]. If the pure ghost Ansatz for the BRST operator in the vacuum suggested by Rastelli, Sen and Zwiebach [8] is correct, solutions to (3.1) will correspond to D-branes condensing out of the vacuum.

Using the operator description (2.29) of string states, solutions to (3.1) simply correspond to projection operators on the space of functionals on the half string. Such projection operators can be classified by their rank r. The identity $\hat{I}$ is an example of an infinite rank projection operator.

In section 3.1 we discuss rank one projection operators on the space of half-string functionals. We formulate conditions on a gaussian state in the full string Fock space which must be satisfied for it to correspond to a rank one projector onto a gaussian state of the half string. In the appendix we show that the D-instanton state related to the sliver satisfies these conditions. Section 3.2 describes a number of ways in which the D-instanton state can be used to construct a large number of other

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projection operators corresponding to other D-instanton and higher-dimensional D-brane states, and addresses the question of when these states are gauge equivalent. In 3.3 we describe multiple D-brane configurations corresponding to higher rank projections.

3.1 Rank one projection operators

The general form of a rank one projector is

$$\hat{\Psi} = |\chi\rangle\langle\chi|\,, \tag{3.2}$$

where $\Psi[x(\sigma)]$ takes the factorized form

$$\Psi[l(\sigma), r(\sigma)] = \chi[l(\sigma)]\chi[r(\sigma)]\,, \tag{3.3}$$

and

$$\chi[l(\sigma)] = \langle l(\sigma)|\chi\rangle\,. \tag{3.4}$$

One simple class of rank one projection operators are described by gaussian functionals

$$\chi[l(\sigma)] \sim \exp\left(-\frac{1}{2} l_{2k+1} M_{kj} l_{2j+1}\right). \tag{3.5}$$

Such a projection operator is described in the full string mode basis by a functional

$$\exp\left(-\frac{1}{2} x_n L_{nm} x_m\right), \tag{3.6}$$

where

$$\begin{aligned} L_{2k+1,2j+1} &= 2M_{kj} \\ L_{2n,2m} &= 2X_{2n,2k+1} M_{kj} X_{2j+1,2n} \\ L_{2k+1,2m} = L_{2n,2j+1} &= 0\,. \end{aligned} \tag{3.7}$$

Turning this discussion around, we can state the conditions on a state of the form (3.6) which are necessary for it to be a projection operator of the form (3.5). To be a projection operator onto a gaussian half-string state, it must be the case that:

(a) The components L_{nm} must vanish when $n+m$ is odd.

(b) The nonzero components must satisfy

$$L_{2n,2m} = X_{2n,2k+1} L_{2k+1,2j+1} X_{2j+1,2m}\,. \tag{3.8}$$

There are some simple functionals which satisfy conditions (a) and (b). For example, from (2.24) we see that

$$L_{nm} = \delta_{nm} \qquad \text{and} \qquad L_{nm} = \delta_{nm} n^2 \tag{3.9}$$

satisfy (3.8) as well as condition (a). The corresponding half-string states are given by (3.5) where

$$M_{kj} = \frac{1}{2}\delta_{kj} \qquad \text{and} \qquad M_{kj} = \frac{1}{2}\delta_{kj}(2k+1)^2 . \tag{3.10}$$

There is a further condition, however, which we would like the state Ψ associated with the projection operator to satisfy, which is that it should be a normalizable state in the Fock space built on the ground state whose wavefunction is

$$\Psi_0[\{x_n\}] \sim \exp\left(-\sum_{n=1}^{\infty} \frac{n}{2} x_n^2 - x_0^2 \right) . \tag{3.11}$$

This leads us to impose the additional condition that the matrix L_{nm} behave like $n\delta_{nm}$ for large n, m. If we describe the projection operator as a squeezed state in the Fock space basis through

$$|\Psi\rangle = e^{-\frac{1}{2} a_n^\dagger S_{nm} a_m^\dagger} |\Omega\rangle , \tag{3.12}$$

then by (2.35) we have

$$L = E^{-1} \left[\frac{1-S}{1+S} \right] E^{-1} . \tag{3.13}$$

The Fock space normalizability condition on L then implies the condition that S have a spectrum with absolute value bounded above by 1. If the eigenvalues of S are s_i we must have $\sum_i s_i^2 < \infty$. This normalizability condition is not satisfied by either of the states defined in (3.9).

In this discussion we have not worried about the overall normalization of states such as (3.2) which is needed for them to satisfy the projection equation (3.1). Even if a state Ψ is normalizable in the Fock space and is described by a functional which can be factorized through (3.3), it may be that an infinite or vanishing normalization factor is needed for (3.1) to be satisfied. As discussed in [9], however, there is an undetermined constant which appears when we separate the matter and ghost sectors. Actually, we only really require that the matter field satisfy

$$\Psi_m = \kappa \Psi_m \star \Psi_m , \tag{3.14}$$

where the ghost field satisfies $Q\Psi_g = \kappa^{-1}\Psi_g \star \Psi_g$ for some κ. In this paper we do not worry about the exact normalization constant κ, which may even formally be 0 or ∞.

To summarize, we have three conditions which a gaussian state of the form (3.5) must satisfy to be a rank one projection operator. The matrix L_{nm} must satisfy conditions (a) and (b), and the associated squeezed state must have a matrix S whose spectrum is such that (3.12) is a finite norm state in the Fock space. It may seem quite difficult to find such a state, since the condition on the spectrum of S and the relation (3.8) are tricky to balance. Nonetheless, it turns out that a class of states related to the sliver state $|\Xi\rangle$ found in [10, 11] satisfy all these conditions.

The sliver state was first constructed as a solution of (3.1) by Kostelecky and Potting in [10]. These authors constructed this state as a zero momentum squeezed state of the form (3.12), and gave a formula for S_{mn} in terms of the coefficients appearing in the Fock space representation of the cubic interaction vertex describing the string field star product. They found numerical evidence that the eigenvalues of S are smaller than one in absolute value and converge to zero as needed for the state to be normalizable in the Fock space.[1] In [9], Rastelli Sen and Zwiebach conjectured that the sliver state corresponds to the space-filling D25-brane in the matter sector of the model they proposed with pure ghost BRST operator. They also constructed a general class of solitonic solutions related to the sliver state but localized in some subset of space-time dimensions. They suggested that these lump solutions correspond to lower-dimensional Dp-branes, and demonstrated that these solutions had the correct tensions relative to the D25-brane solution.

In the appendix, we show that the D-instanton relative $|\Xi_{-1}\rangle$ of the sliver state, which is localized in all space-time dimensions, is a rank one projection operator on the space of half-string states, and we identify the associated functional in terms of half-string degrees of freedom. We expect that all the higher-dimensional lump solutions, including the space-filling sliver state can be similarly described as projection operators which can be written explicitly as functionals of the half-string degrees of freedom. In the next subsection we show how the D-instanton state can be used to construct a range of other rank one projection operators corresponding to D-instantons and higher-dimensional D-branes.

3.2 Other projectors from the D-instanton sliver

From the descriptions of half-string projection operators in terms of full- and half-string modes, we can see that given one projection such as the D-instanton, it is straightforward to construct a wide class of other projections. One way of doing this is to modify a projection operator defined by an L satisfying (3.8) by adding a small shift to L which also satisfies (3.8) but does not violate the normalizability condition. For example, if we start with the D-instanton rank one projector having the L defined through (A.17), (A.2) in the appendix, we can define a projection operator with a different width in space-time through

$$\tilde{L}_{nm} = L_{nm} + \lambda_{nm} , \tag{3.15}$$

[1] Note added: it may be, however, that S nonetheless has an eigenvalue of -1, taking the sliver state slightly outside the Hilbert space [17, 18].

where $\lambda_{00} = \delta$,

$$\lambda_{2k+1,2j+1} = X_{2k+1,0} X_{0,2j+1} \delta = \frac{(-1)^{j+k} 8}{\pi^2 (2k+1)(2j+1)} \delta \,, \tag{3.16}$$

and all other elements of λ vanish. The new operator defined by $\tilde{L}$ is clearly a rank one projection, since (3.8) is still satisfied, and the small change in L for large k, j should not affect the normalizability of the state. By performing this shift with different values of δ in each space-time direction, we can construct a large class of different rank one projection operators corresponding to D-instantons localized at the origin. We expect that all these operators are related to one another by gauge transformations. A similar family of (presumably gauge equivalent) projection operators corresponding to branes with different widths in the transverse directions was described in [9] by using a one-parameter family of creation and annihilation operators to describe the zero-modes x_0.

Starting with the D-instanton, we can modify the projection operator more dramatically by using a shift of the form (3.15) to set all coefficients L_{0n} to zero in $p+1$ of the space-time dimensions $X^{\|}$, shifting the coefficients $L_{2k+1,2j+1}$ as needed to maintain the relation (3.8). The resulting state is not normalizable, but this is because the string functional is independent of $p+1$ of the zero modes $x_0^{\|}$. Dividing the norm of the state by the $p+1$-dimensional volume of space-time in these directions gives us a finite normalization for these states. These projection operators are really projectors of infinite rank on the full space of half-string states, but they are rank one on the space of states independent of $x^{\|}$. We refer to these as "transverse rank one" projection operators. The projection operators constructed in this fashion correspond to higher-dimensional Dp-brane states in the RSZ model. It is not clear whether the functionals describing these states will be identical with the higher-dimensional solutions found in [10, 9], but the only difference for geometrically identical brane solutions should be a nontrivial gauge transformation which may be needed to move from one description to the other.

Another way to construct new rank one projection operators given a single projector onto a gaussian state is to multiply the state by fields which take the left and right parts of the string separately into a new state. For example, given a state described by a functional

$$e^{-\frac{1}{2} l \cdot M \cdot l - \frac{1}{2} r \cdot M \cdot r} \,, \tag{3.17}$$

that satisfies the conditions needed for normalizability in the Fock space, we can construct a projection of the form

$$l_{2k+1} e^{-\frac{1}{2} l \cdot M \cdot l - \frac{1}{2} r \cdot M \cdot r} r_{2k+1} \tag{3.18}$$

that projects onto an orthogonal state from (3.5) for any value of k. In the language of full-string functionals, such a state would be described by

$$\left[\left(\sum_n x_{2n} X_{2n,2k+1}\right)^2 - x_{2k+1}^2\right] \exp\left(-\frac{1}{2} x_n L_{nm} x_m\right), \tag{3.19}$$

where L_{nm} is given by (3.7). This state can be described in the Fock space using (2.37) and (2.2, 2.3). This construction can clearly be generalized by multiplying (3.17) by any set of functions of the form $f_a(\{l_{2k+1}\}) f_a(\{r_{2k+1}\})$, $a = 1, \ldots, r$ and then orthonormalizing to get r independent rank one projectors.

We can of course translate or rotate any of the solutions just described by acting on the zero modes x_0 in the appropriate fashion. This gives us a variety of ways to modify the single projection operator we have described explicitly in the appendix, giving descriptions of single Dp-branes of arbitrary dimension and position in terms of functionals of half-string variables. In the next subsection we describe how these transverse rank one projections can be combined to construct higher rank projection operators corresponding to multiple D-brane states. We conclude this subsection with a discussion of the gauge equivalence of different rank one projectors.

All rank r projection operators on a Hilbert space $\mathcal{H}$ are equivalent to one another under conjugation by a unitarity transformation U

$$\hat{\Psi}_r = U \hat{\Psi}'_r U^\dagger \,. \tag{3.20}$$

In string field theory, such transformations can be realized by exponentiating gauge transformations with gauge parameter $\Lambda_m \otimes I_g$ which leave the ghost part of a tensor product state $\Psi_m \otimes \Psi_g$ unchanged. Naively, this would suggest that all the rank one projection operators we have described here should be gauge equivalent in string field theory. This is contrary to our understanding of the physics, which suggests that Dp-branes located at different points in space-time should not be gauge equivalent. (Of course, a Dp-brane and a Dq-brane for $p \neq q$ are not equivalent since they are described by rank one projection operators on spaces of functionals independent of different sets of zero modes $x^{\|}$.) To understand the resolution of this puzzle regarding two Dp-branes with transverse separation, let us consider a pair of D-instanton states centered at different points 0 and y in space-time. These two states can be written in the full string Fock space as

$$|\Xi_{-1}\rangle, \qquad e^{y\partial/\partial_{x_0}} |\Xi_{-1}\rangle \,. \tag{3.21}$$

In the appendix we show that $|\Xi_{-1}\rangle$ can be described in the split string language as a functional $\chi[l]\chi[r]$ where

$$\chi[\{l_{2k+1}\}] = e^{-l \cdot L \cdot l/4} \tag{3.22}$$

and $L_{2k+1,2j+1}$ is given in (A.17) in terms of (A.2). The generator of a unitarity transformation which takes a projector $|\chi\rangle\langle\chi|$ into an orthogonal projector $|\alpha\rangle\langle\alpha|$ is given by $|\chi\rangle\langle\alpha| - |\alpha\rangle\langle\chi|$. Thus, the matter part of the gauge transformation taking us from the D-instanton centered at the origin to the D-instanton centered at the space-time point y, if it exists, should be given by

$$\Lambda[l,r] = \chi[l]\chi[r+y] - \chi[l+y]\chi[r]\,. \tag{3.23}$$

Let us consider the first term in this expression. Translating this back into the language of the full string Fock space, $\chi[l]\chi[r+y]$ becomes

$$\exp\left(\frac{y}{2}\frac{\partial}{\partial_{x_0}} - \sum_k \frac{y}{2} X_{0,2k+1}\frac{\partial}{\partial_{x_{2k+1}}}\right)|\Xi_{-1}\rangle\,. \tag{3.24}$$

While both states in (3.21) are normalizable states in the Fock space, the same is not true of (3.24). This state has a unit norm but has a vanishing inner product with all the states in the Fock space. This result essentially follows from the fact that the shifts in (3.24) have the effect of breaking the string so that the left and right halves of the string do not have the same value at $\sigma = \pi/2$. This means that the resulting functional has vanishing overlap with states in the Fock space, which are localized around smooth string configurations. To see this more concretely, consider the related state

$$\exp\left(\frac{y}{2}\frac{\partial}{\partial_{x_0}} - \sum_k \frac{y}{2} X_{0,2k+1}\frac{\partial}{\partial_{x_{2k+1}}}\right)|\Omega\rangle\,. \tag{3.25}$$

Rewriting the derivatives in terms of creation and annihilation operators we have

$$\exp\left[i\frac{y}{2}(a_0 + a_0^\dagger) - i\frac{y}{2}\sum_{k=0}^{\infty}\frac{(-1)^k 2}{\pi\sqrt{2k+1}}(a_{2k+1} + a_{2k+1}^\dagger)\right]|\Omega\rangle\,. \tag{3.26}$$

Normal ordering gives us

$$\exp\left[i\frac{y}{2}a_0^\dagger - i\frac{y}{2}\sum_{k=0}^{\infty}\frac{(-1)^k 2}{\pi\sqrt{2k+1}}a_{2k+1}^\dagger - \frac{y^2}{8}\left(1 + \sum_k \frac{4}{\pi^2(2k+1)}\right)\right]|\Omega\rangle\,. \tag{3.27}$$

The infinite negative constant indicates that the overlap of this state with any state in the Fock space vanishes. On the other hand, taking the inner product of this state with itself we see that the normal ordering constant cancels, so that this state has norm one (which is expected since it is simply a translation in an infinite number of modes from the original state). This shows that the state does not lie in the usual Fock space. Since the state $|\Xi_{-1}\rangle$ is in the Fock space, the same argument shows that (3.24) is not in the Fock space. Since this state is not in the Fock space, it cannot be used to generate a gauge transformation in the matter sector. This explains why the states (3.21) are not gauge equivalent.

In this subsection we have constructed a family of rank one projection operators onto half-string functionals independent of $p+1$ space-time directions $x^{\|}$. These projection operators correspond in the BPZ model to matter sector descriptions of Dp-branes of arbitrary dimension, position and orientation. In the following subsection we discuss how these projections may be combined to form higher rank projections describing arbitrary multiple D-brane configurations.

3.3 Higher rank projections and multiple D-branes

In general, projection operators on the space of half-string functionals

$$\hat{P}_1 = |\chi\rangle\langle\chi|\,, \tag{3.28}$$

that are rank one on the space of functionals independent of some set of space-time directions $x^{\|}$ will correspond to single D-brane configurations, when the matter sector of the string field theory is decoupled as in the RSZ model [8]. We have identified a particular class of rank one projection operators as functionals on the full string satisfying certain conditions.

We can combine r mutually orthogonal transverse rank one projection operators into a projection operator of the form

$$\hat{P}_r = |\chi_1\rangle\langle\chi_1| + |\chi_2\rangle\langle\chi_2| + \cdots + |\chi_r\rangle\langle\chi_r|\,, \tag{3.29}$$

where $\langle\chi_i|\chi_j\rangle \sim \delta_{ij}$. These operators correspond to configurations of r distinct D-branes, which may be coincident or separated in transverse space-time dimensions. In the previous subsection we described a number of ways of constructing projection operators describing single Dp-branes of different dimensions. Other constructions of single Dp-brane projection operators were given in [9]. By choosing a set of r orthogonal projectors constructed in any of these ways, we can construct a projector through (3.29) describing a system of r D-branes. This allows us in principle to construct any multiple D-brane configuration as a projection operator in the matter sector of the string field theory.

Just as some pairs of rank one projectors are gauge equivalent and others, which correspond to physically distinct D-branes, are not gauge equivalent, some rank r projectors are gauge equivalent and others are not. An example of interest is given by a set of r orthogonal transverse rank one projectors corresponding to zero momentum states in the full string Fock space. As discussed in subsection (2.4), when we consider the state Ψ as a function of space-time taking values in the space of operators on the ND string Hilbert space $\mathcal{H}$, zero momentum states are identified with single operators on $\mathcal{H}$, so the r projectors are each rank one when considered as operators on $\mathcal{H}$. We expect all states describing r D25-branes to be gauge equivalent, which corresponds to the fact that any rank r projector on $\mathcal{H}$ can be related to any other rank r projector by a unitarity transformation on $\mathcal{H}$ which can be realized as a gauge transformation in string field theory parameterized by a zero momentum field.

Let us now describe the form of the potential around a "transverse rank r" projection operator describing r coincident D-branes, to verify that there are r^2 tachyonic modes as we expect from perturbative string theory. If we write a basis for the space of half-string functionals independent of some space-time coordinates $x^{\|}$ as $|\chi_i\rangle, i \in \mathbb{Z}$, then for a state described by an operator which is a sum of projections of the form

$$\hat{\Psi} = \sum_i |\chi_i\rangle \psi_i \langle\chi_i| \,, \tag{3.30}$$

the string field potential will be (removing an overall factor of $\int \Psi_g \star Q \Psi_g + \Psi_g \star \Psi_g \star \Psi_g$)

$$V = \mathrm{Tr}\left[\frac{1}{2}\hat{\Psi}^2 - \frac{1}{3}\hat{\Psi}^3\right] = \sum_i \frac{1}{2}\psi_i^2 - \frac{1}{3}\psi_i^3 \,. \tag{3.31}$$

The state (3.30) solves the equations of motion if $\psi_i \in \{0, 1\}$ $\forall i$. The rank r projections have r nonzero values of ψ_i and can be written in the form (3.29) by a relabeling of the basis. If we expand the full $x^{\|}$-independent string field in the split string language

$$\hat{\Psi} = \sum_{i,j} |\chi_i\rangle \psi_{ij} \langle\chi_j| \,, \tag{3.32}$$

then we can calculate the quadratic terms in the action around the rank r projection (3.29). These terms are given by

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{2}\psi_{ij}\psi_{ji} - \sum_{i=1}^{r}\sum_{j=1}^{\infty} \psi_{ij}\psi_{ji} \,. \tag{3.33}$$

For fields Ψ_{ij} with $i, j > r$ we have a positive quadratic term. These are massive degrees of freedom. For fields with one index $\leq r$ and the other $> r$ we have vanishing quadratic terms. These fields correspond to gauge degrees of freedom which rotate one form of the projection operator into another, as discussed in the previous subsection. The N^2 fields with $1 \leq i, j \leq r$ all have negative quadratic terms. These correspond to the r^2 tachyonic open string fields on a system of r unstable D25-branes. Note that if the branes we considered here were not coincident, the operators $|\chi_i\rangle\langle\chi_j|, i \neq j$ would not lie in the full string Hilbert space, like the gauge transformation parameter (3.24), so we would only have r tachyonic modes, as we expect for r separated Dp-branes.

4. Conclusions

In this paper we have shown that in the matter sector of Witten's cubic open string field theory it is possible to describe string fields as operators on the space of functionals associated with the degrees of freedom on a half string. These functionals were described in terms of half-string oscillator modes. We identified rank r projection operators as configurations of r D-instantons in the Rastelli-Sen-Zwiebach model where the BRST operator Q is taken to be pure ghost, and rank r projection operators on functionals independent of $p+1$ zero modes as configurations of r Dp-branes. We showed that rank one projection operators onto gaussian states of the half-string could be described as gaussian states of the full string satisfying certain conditions. We showed that a particular D-instanton state constructed by Rastelli, Sen, and Zwiebach satisfies these conditions and can be identified as a rank one projection. We showed how this state can be modified to describe a Dp-brane of arbitrary dimension, and that multiple such states can be added to produce an arbitrary configuration of D-branes. Finally, we showed that the number of tachyonic degrees of freedom around a rank r projector describing r coincident D-branes is r^2, in agreement with the expectation from perturbative string theory.

In order to really understand the structure of the vacuum in cubic string field theory, either with the standard perturbative string BRST operator Q or with a ghost BRST operator as suggested by RSZ, it is clearly essential to develop an analogous analytic approach to describing the ghost sector of the theory. The most difficult part of representing the ghost sector in terms of half-string degrees of freedom is the representation of the ghost number one BRST operator in terms of the half-string degrees of freedom. In a sequel to this paper we will treat this problem and describe the ghost sector of the theory in a similar fashion to the way in which we have described the matter sector of the theory in this work.

A. The D-instanton sliver as a rank one projector

In this appendix we prove that a particular solution of the matter string field equation of motion $\Psi = \Psi \star \Psi$ which was found in [10, 9] is a rank one projection operator onto a gaussian state on the half string. In [10], Kostelecky and Potting found a zero-momentum solution of the equations of motion by making the assumption that the solution took the form of a particular kind of squeezed state and then by solving the equation of motion explicitly using the mode representation of the three string vertex. Later, Rastelli, Sen and Zwiebach identified the state found by Kostelecky and Potting as the sliver state found in [11]. These authors also found a more general class of solutions related to the sliver state but with gaussian dependence on space-time zero modes. They identified these states as Dp-branes with $p < 25$, and showed that the ratio of tensions between a pair of Dp-branes of different dimensions is correctly reproduced by these gaussian states. The precise state we consider here is a squeezed state which is localized in all space-time dimensions, namely the D-instanton lump solution constructed by RSZ. We show that this state is a rank one projection on the half string using the formalism developed in the text. The connection of other higher-dimensional Dp-brane solutions to this D-instanton is discussed in the main text.

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We begin by recalling some useful formulae from [13] and the definition of the D-instanton state following [10, 9]. The D-instanton state is a state of the form

$$|\Xi_{-1}\rangle = e^{-\frac{1}{2}a_n^\dagger S_{nm} a_m^\dagger}|\Omega\rangle\,, \tag{A.1}$$

with

$$S = CT\,, \tag{A.2}$$

where C is defined in (2.11) and

$$T = \frac{1}{2Z}\left(1 + Z - \sqrt{(1+3Z)(1-Z)}\right). \tag{A.3}$$

(Note that in [10, 9], Z is called X, but this conflicts with the notation used here and in [13] for X). The matrix Z is given by

$$Z = CV\,, \tag{A.4}$$

where V is the matrix U^{11} appearing in [13] in the cubic string vertex constructed using creation-annihilation operators $a_0, a_0^\dagger$. (Note that there are several different notations used in the literature for the matrices U, V we consider here. The matrices we call U, V are the same as U, U^{11} in [13], U', V' in [10], and U^{gj}, V' in [9].) The matrix V can in turn be related to the change of basis matrix X defined in (2.23) by writing

$$V = \frac{1}{3}\left(C + U + \bar{U}\right) \tag{A.5}$$

where

$$U = \left(2 - EYE^{-1} + E^{-1}YE\right)\left[EYE^{-1} + E^{-1}YE\right]^{-1}, \tag{A.6}$$

E is defined in (2.5), and

$$Y = -\frac{1}{2}C - \frac{\sqrt{3}}{2}iCX\,. \tag{A.7}$$

From these definitions we can derive some useful relations. We have

$$Y^2 = 1\,. \tag{A.8}$$

From this we can derive

$$\begin{aligned}(1-Y)E(1+U) &= 0\\ (1+Y)E^{-1}(1-U) &= 0\,.\end{aligned} \tag{A.9}$$

We will find it useful to decompose all these matrices into two by two block matrices associated with odd and even indices, as is done in (2.23). We thus have

$$X = \begin{pmatrix} 0 & X_{oe} \\ X_{eo} & 0 \end{pmatrix}, \tag{A.10}$$

where for example $X_{oe} = (X_{2k+1,2m})$. Similarly, we have

$$\begin{aligned}
C &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\
U &= \begin{pmatrix} U_{oo} & U_{oe} \\ U_{eo} & U_{ee} \end{pmatrix}, \qquad \bar{U} = \begin{pmatrix} U_{oo} & -U_{oe} \\ -U_{eo} & U_{ee} \end{pmatrix}, \\
V &= \begin{pmatrix} V_{oo} & 0 \\ 0 & V_{ee} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1+2U_{oo} & 0 \\ 0 & 1+2U_{ee} \end{pmatrix}, \\
Z &= \begin{pmatrix} -V_{oo} & 0 \\ 0 & V_{ee} \end{pmatrix}.
\end{aligned} \tag{A.11}$$

Using this block notation, we can use (A.9) to derive some further relations on U_{oo}, U_{ee} which will be useful later. From the oo, oe components of the two relations in (A.9) we have

$$\begin{aligned}
(1+U_{oo}) - \sqrt{3} i M U_{eo} &= 0\,, \\
-\sqrt{3} i M (1+U_{ee}) + U_{oe} &= 0\,, \\
3(1-U_{oo}) - \sqrt{3} i (M^T)^{-1} U_{eo} &= 0\,, \\
\sqrt{3} i (M^T)^{-1} (1-U_{ee}) - 3U_{oe} &= 0\,,
\end{aligned} \tag{A.12}$$

where we have defined

$$M = E^{-1} X_{oe} E\,, \qquad (M^T)^{-1} = E X_{oe} E^{-1}\,. \tag{A.13}$$

From (A.12) we have

$$\begin{aligned}
1+U_{oo} &= 3M(1+U_{ee})M^T \\
1-U_{oo} &= \frac{1}{3}(M^T)^{-1}(1-U_{ee})M^{-1}\,.
\end{aligned} \tag{A.14}$$

From these we have

$$1 - U_{oo}^2 = (M^T)^{-1}(1-U_{ee}^2)M^T\,, \tag{A.15}$$

which implies

$$\sqrt{1-U_{oo}^2} = (M^T)^{-1}\sqrt{1-U_{ee}^2}M^T\,. \tag{A.16}$$

We now wish to use these relations to demonstrate that $|\Xi_{-1}\rangle$ is a projection operator on the space of half-string functionals. Since T is a function of Z, and Z_{nm} vanishes when $n+m$ is odd, condition (a) from section 3.1 is clearly satisfied. It remains to demonstrate that condition (b) is satisfied. Condition (b) states that the matrix

$$L = E^{-1}\left(\frac{1-S}{1+S}\right)E^{-1}\,, \tag{A.17}$$

satisfies

$$L_{ee} = X_{eo} L_{oo} X_{oe}\,. \tag{A.18}$$

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It is reasonable that such a condition should be satisfied, since as we have seen L is defined as a function of X. Let us now verify this explicitly.

From (A.3) we have

$$S = CT = \begin{pmatrix} \frac{1}{2V_{oo}}\left[1 - V_{oo} - \sqrt{(1-3V_{oo})(1+V_{oo})}\right] & 0 \\ 0 & \frac{1}{2V_{ee}}\left[1 + V_{ee} - \sqrt{(1+3V_{ee})(1-V_{ee})}\right] \end{pmatrix}. \tag{A.19}$$

This gives us

$$1+S = \begin{pmatrix} \frac{1}{2V_{oo}}\sqrt{1+V_{oo}}\left[\sqrt{1+V_{oo}} - \sqrt{1-3V_{oo}}\right] & 0 \\ 0 & \frac{1}{2V_{ee}}\sqrt{1+3V_{ee}}\left[\sqrt{1+3V_{ee}} - \sqrt{(1-V_{ee})}\right] \end{pmatrix} \tag{A.20}$$

and

$$1-S = \begin{pmatrix} \frac{1}{2V_{oo}}\sqrt{1-3V_{oo}}\left[\sqrt{1+V_{oo}} - \sqrt{1-3V_{oo}}\right] & 0 \\ 0 & \frac{1}{2V_{ee}}\sqrt{1-V_{ee}}\left[\sqrt{1+3V_{ee}} - \sqrt{(1-V_{ee})}\right] \end{pmatrix}, \tag{A.21}$$

so that

$$L = E^{-1}\begin{pmatrix} \frac{\sqrt{1-3V_{oo}}}{\sqrt{1+V_{oo}}} & 0 \\ 0 & \frac{\sqrt{1-V_{ee}}}{\sqrt{1+3V_{ee}}} \end{pmatrix} E^{-1}. \tag{A.22}$$

Using (A.11) this becomes

$$L = E^{-1}\begin{pmatrix} \sqrt{3}\frac{\sqrt{1-U_{oo}}}{\sqrt{1+U_{oo}}} & 0 \\ 0 & \frac{1}{\sqrt{3}}\frac{\sqrt{1-U_{ee}}}{\sqrt{1+U_{ee}}} \end{pmatrix} E^{-1}. \tag{A.23}$$

To show that $|\Xi_{-1}\rangle$ is a projector, we must prove (A.18), which from (A.23) means that we must have

$$3M^T\sqrt{\frac{1-U_{oo}}{1+U_{oo}}}M = \sqrt{\frac{1-U_{ee}}{1+U_{ee}}}, \tag{A.24}$$

where M is defined in (A.13). We can rearrange this equation into the form

$$3\frac{1}{\sqrt{1-U_{ee}}}M^T\sqrt{1-U_{oo}} = \frac{1}{\sqrt{1+U_{ee}}}M^{-1}\sqrt{1+U_{oo}}. \tag{A.25}$$

This leads to

$$3\frac{1}{\sqrt{1-U_{ee}^2}}M^T\sqrt{1-U_{oo}^2} = \frac{1}{1+U_{ee}}M^{-1}(1+U_{oo}), \tag{A.26}$$

which follows directly from (A.14) and (A.16).

We have now shown that the state $|\Xi_{-1}\rangle$, corresponding to a D-instanton in the RSZ model, satisfies the relations (a) and (b) from section 3.1 which are needed for a gaussian functional to act as a rank one projection on the space of half-string functionals. For the state $|\Xi_{-1}\rangle$ to be a normalizable state in the Fock space, it is furthermore necessary that the eigenvalues of S be less than one and be square summable. This in turn requires that the spectrum of Z lie in the range $(-1/3, 1)$ and approach zero. This condition was checked numerically for the sliver state in [10]. We have verified numerically that the spectrum of Z for the D-instanton state $|\Xi_{-1}\rangle$ is similar to that of the sliver state and seems to be such that this state also is normalizable and in the Fock space.[2] Thus, we have found that the state $|\Xi_{-1}\rangle$ corresponds to a normalizable state in the string Fock space which acts as a rank one projection on half-string functionals.

Acknowledgments

Thanks to M. Douglas, G. Moore, L. Rastelli, A. Sen, I. Singer, and B. Zwiebach for helpful discussions and correspondence. WT would like to thank the ITP, Santa Barbara, and the ITP workshop on M-theory for support and hospitality during the progress of this work. The work of DJG was supported by the NSF under the grants PHY 99-07949 and PHY 97-22022. The work of WT was supported in part by the A.P. Sloan Foundation and in part by the DOE through contract #DE-FC02-94ER40818.

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